ENGINEERING DIVISION

Tolerances for PAL colour television

TECHNOLOGICAL REPORT No. GE-1

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THE BRITISH BROADCASTING CORPORATION
ENGINEERING DIVISION

RESEARCH DEPARTMENT

TOLERANCES FOR PAL COLOUR TELEVISION

Technological Report No. GE-1
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TOLERANCES FOR PAL COLOUR TELEVISION

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September 1967.

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TOLERANCES FOR PAL COLOUR TELEVISION

This report is based upon a paper that was read at the XIVth International Scientific Congress on Electronics in Rome, June 1967

SUMMARY

The problem of ensuring satisfactory colour pictures in the viewer's home differs in difficulty but not in kind from that of ensuring satisfactory monochrome pictures. Historically, monochrome picture quality has improved slowly over the years, but we still lack the necessary administrative means by which adequate quality can be guaranteed to the domestic viewer. The sudden addition of colour to an existing monochrome service will, it is thought, not be granted an initial period of grace, by the viewer, during which colour picture quality might be of an inferior grade.

If colour television is to be an immediate success, some administrative means of guaranteeing satisfactory picture quality may have to be established. It is the purpose of this report to suggest the basic engineering lines upon which an administrative superstructure could be built.

Adequate technical information is still lacking, but the major distortions that are introduced by equipment in the colour television chain are known and the amounts of them that are tolerable are likewise known. Thus it is possible to establish overall tolerances which must not be exceeded if colour picture quality is not to fall below a given subjective criterion. Having established overall tolerances, the next problem is to divide each overall tolerance amongst the various portions of the television chain. When this is done it becomes evident that various sections of the television chain would be restricted to tolerances which they cannot maintain. In other words, the arithmetic sums of the performance figures for the individual parts of the chain exceed the permissible overall tolerances. This means that either colour television is impossible or one must admit that the overall tolerances will be exceeded from time to time.

It is fortunate that the errors in the parameters most likely to cause distortion of the colour television picture vary statistically in a positive and a negative direction. Thus the arithmetic sum of all positive errors or of all negative errors is a statistically rare event and convolution of the errors contributed by the various parts of a television chain shows that overall tolerances will be exceeded for only small percentages of the time.

The report gives some examples of a possible subdivision of the colour television chain and of the results of convolution of errors.

1. INTRODUCTION

Although much of what follows could have been applied to monochrome television it was less urgent than for colour, since monochrome television was, it is thought, such a novel experience that the viewing public, as it slowly grew in size, was prepared to accept a technical quality of picture that left much to be desired. Even today, the variations from programme to programme in the technical quality of the images are very great indeed and it is clear that, in general, the broadcasting of television is not a "quality-controlled process" in the industrial sense of the phrase.

Colour, on the other hand, is merely an additional feature, albeit of great value, to the existing monochrome service and it may well be that the viewing public will not be prepared to give it a lengthy period of grace during which colour quality might be variable and at times not fully acceptable. It must be remembered, moreover, that the price of a colour receiver in Europe will be more within the the motor-car range than in the domestic appliance bracket.

How, then, can an adequate colour picture quality be ensured? The major distortions that can arise are known and it is also known how to correct them, but the statistically unpredictable is not known. All that can be done is to try to ensure that the sum of all the statistical errors that give rise to distortions of the viewer's picture is restricted to a value that will maintain viewer satisfaction and not hinder the sale of receivers.

An important administrative difficulty would seem to be the fact that in many countries a number of different organizations, authorities and administrations owing no allegiance to one another contribute each its own quota of distortion to the final product. Some motor-car manufacturers, for example, assemble into their cars components that they, in turn, have bought from other manufacturers, and they, the car makers, vary in their willingness to accept responsibility for these components in cases of unsatisfactory performance. It is doubtful whether satisfactory petrol could be produced from a refinery in which the technical processes were split between two or three separate chemical and oil companies!

2. THE COLOUR TELEVISION CHAIN

Fig. 1 shows a simplified diagram of a colour transmission. It is beyond the scope of this report to delve more deeply into each of the boxes shown in Fig. 1. Suffice it to say

(i) that in the case of international exchange of programmes several complete networks may be connected in tandem, not infrequently with one or more standards converters in between each network;

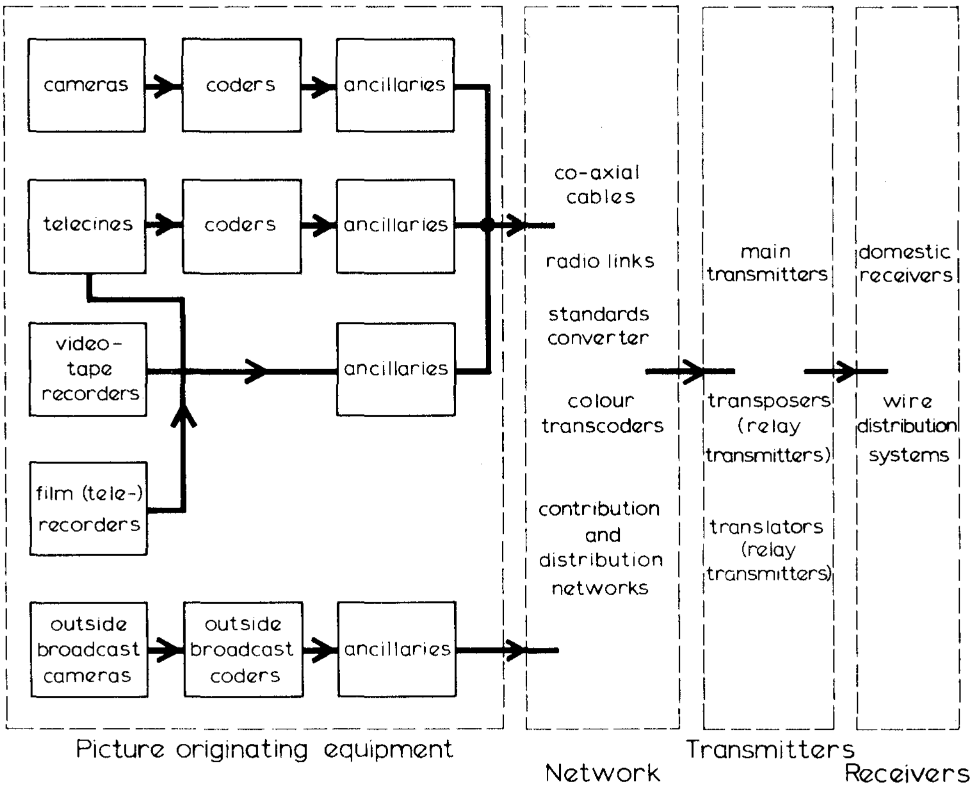

Fig. 1 - A television chain

(ii) that up to about three relay transmitters may follow one main transmitter;

(iii) that a signal passed from its originating point to a contribution network may pass through a central studio complex before reaching the distribution network;

(iv) that the vagaries of propagation, which can be overriding, will not be discussed in this report.

It is a matter of surprise that such relatively satisfactory transcontinental exchanges take place in monochrome. What has already been achieved in colour during the European Broadcasting Union's (EBU's) investigation of the competing colour systems is quite remarkable when the difficulties are realized. Maintaining acceptable colour quality in day-to-day service, however, will demand, in the absence of a common authority, a very considerable effort of application by all concerned.

3. OVERALL TOLERANCES REQUIRED IF VIEWER SATISFACTION IS TO BE MAINTAINED

It was Georges Hansen, Technical Director of the EBU, who, at the beginning of the European colour systems competition, insisted to the chairman of one of the sub-groups of the Ad-hoc Group on Colour Television chaired by Prof. Dr. Theile that an important method of rating the merits of a colour system would be the objective quantity of a given distortion required to degrade the picture quality by an agreed subjective amount. In other words, according to this single criterion, the best colour system would have the largest overall tolerance values in relation to the distortions arising in practice.

An overall tolerance in accordance with the foregoing principle is quite independent of the performance of equipment. It is a function of the principles on which the system is based. It is, therefore, relatively easy to measure. It is necessary only to collect a statistically adequate number of test observers; to seat them in front of a colour display and to distort, in controlled conditions, the colour signal in the way required for the particular distortion under investigation and by various amounts, both more than and less than that required to make each observer give the desired criterion of either picture quality or impairment. The scales of subjective criteria used by the EBU are shown in Table 1. These scales are not intended for the scientific discovery of properties of human senses. They are intended to assist in the rather commercial job of determining whether the product is "saleable" or not.

TABLE 1

EBU Scales of Subjective Criteria

Subjective picture

Quality	Mark	Impairment
Excellent	1A	Imperceptible
Good	2A	Just perceptible
Fairly good	3A	Definitely perceptible but not disturbing
Rather poor	4A	Somewhat objectionable
Poor	5A	Definitely objectionable
Very poor	6A	Unusable

As the result of many tests done in a number of countries, the EBU has published[1] a set of overall tolerances, Table 2, for subjective criteria 2·5A and 3·5A. A rough and ready rule is that, for an acceptable colour service, grade 3·5A is tolerable if only one distortion is present, but when several distortions co-exist simultaneously each one, alone, should not be worse than grade 2·5A.

TABLE 2

EBU Overall Tolerances

Type of receiver	Chrom. signal phase errors		Chrom. signal diff. phase		Amplitude ratio chrom. to lum. signal		Chrom. signal diff. gain		Timing errors chrom. to lum. signal	
	grade		grade		grade		grade		grade	
	2·5A	3·5A	2·5A	3·5A	2·5A	3·5A	2·5A	3·5A	2·5A	3·5A
PAL_d PAL_n	±40°	±50°	±40°	±50°	±2½ dB	±4 dB	30%	40%	+200 ns −300 ns	+350 ns −500 ns
PAL_s	±12°	±15°	±12°	±20°	±2½ dB	±4 dB	30%	40%	+200 ns −300 ns	+350 ns −500 ns

There is, of course, a great number of parameters which, if transmitted with sufficient error, will contribute to the degradation of picture quality. This report is, of necessity, restricted to those which apply only to colour and a further restriction is imposed to consideration of those colour parameters whose errors have been adequately assessed. One is thus left with differential gain, differential phase, timing of the chrominance signal relative to the luminance signal and ratio of amplitudes of chrominance signal to luminance signal. If this list is inadequate, it certainly includes some important parameters!

A rather *faute de mieux* decision by the EBU was that these overall tolerances should be applicable only to that part of the colour television chain between coder and decoder. It was felt that processes taking place on the three primary-colour signals, particularly at the picture originating end of the chain, might be difficult to distinguish from the artistic side of the colour programme. Moreover, it was felt that the specification of tolerances for cameras, telecine machines and similar equipment was more a question for individual negotiations between television studio operators and the equipment manufacturers rather than for administrations and broadcasters. It must be emphasized that a great deal of work is necessary on the establishment of tolerances and the standardization of performance of the equipment lying outside these limits, that is, in the programme generating equipment, the recording and reproducing equipment, and also in the final receiving and display system. This report, however, relates essentially to the summation of the distortions arising between the stated limits. The principles should, of course, be extended to cover the wider aspects.

4. PERFORMANCE FIGURES OF TELEVISION EQUIPMENT AND MAINTENANCE OF OVERALL TOLERANCES

The EBU divided the television chain into five sections for the purpose of elucidating the relevant performance figures of the equipment. Fig. 2 shows these subdivisions and a suggested simplification; namely, the grouping of the video-tape recorder with the studio complex. This would accord better with Fig. 1 in which an attempt has been made to group together only equipment under single ownership. The reverse is often not true; that is, one owner may well possess more than one group, but administratively this would not seem to matter.

The individual performance figures established by the EBU were defined rather loosely as figures representative either of normal practice or of the best attainable at the time the figures were written down - in the expectation that improvements in technique would render them achievable in normal operating conditions in the near future. Table 3 shows these figures for the five portions of the chain. They are just bare numbers. Will they be exceeded and, if so, for what proportion of the time? How will they add, in practice? Will their sums exceed the overall tolerances?

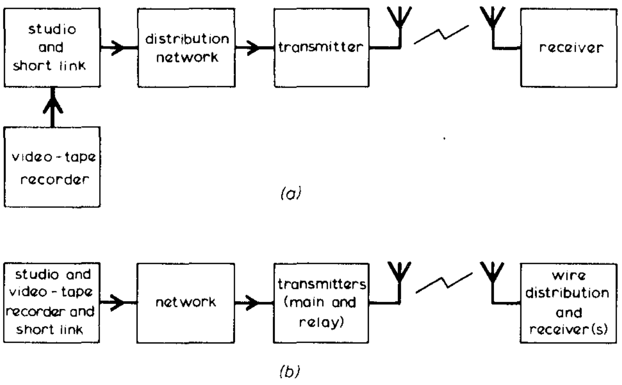
studio and short link
distribution network
transmitter
receiver
video-tape recorder
(a)
studio and video-tape recorder and short link
network
transmitters (main and relay)
wire distribution and receiver(s)
(b)

Fig. 2 - Subdivision of television chain

(*a*) EBU subdivision of chain
(*b*) Suggested subdivision of chain

5. CONVOLUTION OF ERRORS

The application of convolution to aperture theory in physics is perhaps better known than its application to the statistical theory of the addition of errors. If a one-dimensional optical aperture has an optical transmission distribution function, $f(x)$, where x is the unique dimension, and this aperture is then scanned by an active one whose brightness distribution is $g(x)$, the function resulting from the scanning of $f(x)$ by $g(x)$ is

$$h(x) = \int_{-\infty}^{\infty} f(\lambda)g(x-\lambda)d\lambda$$

Similarly, if the response of an electric circuit to a unit-impulse excitation (an excitation that examines the performance of the circuit at all frequencies) is $f(t)$ and if this circuit be excited by an excitation function $g(t)$, the response, $h(t)$ will be

$$h(t) = \int_{-\infty}^{\infty} f(\lambda)g(t-\lambda)d\lambda$$

The meaning of convolution is clearer in connection with the addition of errors and its arithmetic form will now be used. Consider two amplifiers in tandem connection. Suppose that the gain of each is supposed to be 0 dB, but that the gains

TABLE 3

EBU Performance Figures

Portion of colour television chain	Chrom. signal phase errors	Chrom. signal diff. phase	Amplitude ratio chrom./lum. signal	Chrom. signal diff. gain	Timing errors chrom. to lum. signal
Receiver	-	±5°	-	10%	?
Transmitter	?	±5°	±1 dB	10%	-
Distribution network (Note)	-	±10°	±2 dB	8%	(±?) 25 ns
Studio centre	?	±3°	?	5%	± 10 ns
Video-tape recorder	±5°	±4°	?	2%	?

Note: Applies only to a video section. The figures are not applicable to a more complex network composed of several "video-to-video" sections in cascade.

- : unimportant or of negligible value or easily adjustable or irrelevant

? : insufficient evidence available

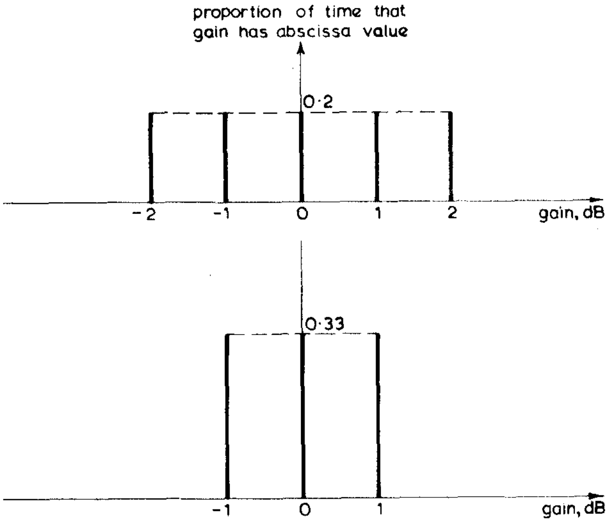

Fig. 3 - Statistical distributions of amplifier gain

of each vary in a random manner, each amplifier giving rise to a rectangular statistical distribution of errors of gain. Fig. 3 shows the two distributions and for simplicity these have been approximated by five equi-spaced samples for one of the amplifiers and three equi-spaced samples for the other. The areas under the rectangles are, for convenience, normalized to unit value. First, we note that the following values of overall gain can occur: −3 dB, −2 dB, −1 dB, 0 dB, 1 dB, 2 dB and 3 dB. Not each one of these values is equally likely to occur, however. The −3 dB value can only occur by the combination of −2 dB with −1 dB and the probability of occurrence of −2 dB from one amplifier is 0·2 whilst the probability of occurrence of −1 dB from the other is 0·33. Thus the probability of the simultaneous occurrence of these two values of gain is 0·066. We now effect the multiplication indicated in Table 4. This process is permutable and may include any number of multiplications. This is the only method of finding the overall statistical distribution of additive errors. The integral or progressive sum of the statistical distribution arrived at by convolution is the cumulative probability curve from which the probability that any given overall error will be exceeded can be read.

TABLE 4

Overall Statistical Distribution of Gain for Two Amplifiers in Tandem

Overall Gain:	−3	−2	−1	0	1	2	3	
		·2	·2	·2	·2	·2		
			·33	·33	·33			
	·066	·066	·066	·066	·066			
		·066	·066	·066	·066	·066		
			·066	·066	·066	·066	·066	
	·066	·132	·198	·198	·198	·132	·066	TOTAL = 1

6. POSSIBLE LAWS OF STATISTICAL FREQUENCY DISTRIBUTIONS OF ERRORS

There appear to be two practical laws. They are rectangular and Gaussian or Normal. It must be remembered that all systematic errors must be corrected so that we shall be considering only random or aleatory errors. The Normal law is convenient because copious tables of its integral, the Error Function, exist. On the other hand it is manifestly false, because the boundaries of its aleatory variable are ± ∞. One is then led to consider truncated Gaussian functions and this is regarded as a nuisance that is not worth while. The rectangular distribution, on the other hand, cannot be convolved by simply adding variances (square of standard deviation) and reading from straight lines drawn on probability graph paper. It may be objected, further, that all values of error within a given range of plus and minus values are not likely to be equally probable. Be that as it may, until the engineers concerned with the equipment contained in the various portions of the television chain produce statistically valid performance figures it would seem that the least controversial law of errors to adopt would be the rectangular form and this is what has been done in this report. It is furthermore assumed that the boundaries of the rectangles that are centred on zero error coincide with the plus and minus values of the EBU performance figures. It is interesting to note that the difference between the cumulative probability of occurrence of errors within a given range, after convolving five rectangular statistical distributions and that obtained after convolving five Gaussian statistical distributions is rather small. Fig. 4 shows such a comparison. Although the rectangular and Gaussian distributions that are convolved to produce Fig. 4 have standard deviations based on the performance figures for differential gain shown in the fifth column of Table 3, this fact need not concern us yet. Consequently the ordinates in Fig. 4 are simply marked "error", rather than "differential gain error in percent." Of course, the smaller the number of rectangular distributions to be convolved, the greater the departure from the Gaussian form. The convolution of only two rectangular distributions, for example, results in a triangular distribution.

A word about power-law addition of errors! CCIR Recommendation 451 suggests this form of error addition and various indices for the power law are suggested for different types of distortion encountered in long-distance television transmission networks. Square law and three-halves power law are mentioned. These are special cases of the rth half-moment of the statistical distribution of the errors of a transmission network or tandem connection of a multiplicity of electronic devices. The more general case involving the use of statistical moments is touched upon, briefly, in the appendix. It is, nevertheless, helpful, in bringing to light the basic difference between a statistical analysis of

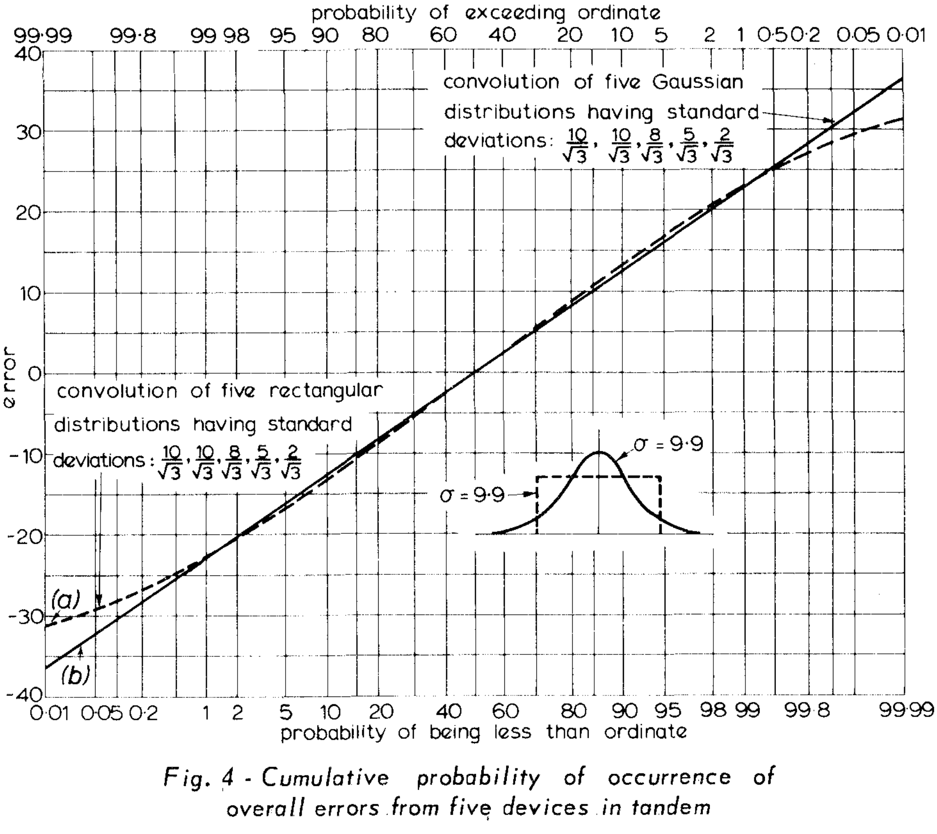

Fig. 4 - Cumulative probability of occurrence of overall errors from five devices in tandem

errors on the one hand and simple power-law addition on the other, to consider how an overall tolerance might be established using each of the two methods. If we have available curve (*a*) (for example) in Fig. 4, we can decide on a tolerance that will not be exceeded (in both its plus and minus values) for more than a given percentage of time, as has been adopted by the C.C.I.R. with regard to co-channel interference with broadcasting. For example, curve (*a*) shows that an error numerically greater than 17 will not occur for more than 10% of the time, (probabilities of 5% and 95% in Fig. 4). On the other hand, if we do not wish to tolerate an error numerically greater than 10 for more than 10% of the time, curve (*a*) shows clearly that such a circumstance is impossible unless some or all of the individual or component rectangular distributions are reduced in width. The application of the power-law addition method of determining tolerances consists in raising each component tolerance to the power r and taking the rth root of the sum. The resulting tolerance can be a rather arbitrary figure that has no calculated connexion with any percentage of time during which it may be exceeded. This disadvantage is the inevitable result of a lack of data of the statistical variations of the errors of the component parts of the television chain and of the lack of fit of the methods of measurement of errors to the requirements of statistical analysis. Table 5 shows, in the second column, tolerances obtained by power-law addition (with $r = 3/2$) from various combinations of four or five component parts of a television chain. The third column shows the percentages of time for which these tolerances will be exceeded. The percentage for the last line of the Table was obtained by applying the power-law-addition tolerance to the ordinate scale of curve (*a*) in Fig. 4 and reading the appropriate probabilities from the scale of abscissae. For the six lines above the last, the percentages of probability (or time) were obtained from cumulative curves (like curve (*a*), Fig. 4) of convolved rectangles indicated in column 1 of the Table.

Although the power-law-addition tolerances maintain an almost constant proportionality to the standard deviations, σ, of the various overall statistical distributions, the probabilities that they will be exceeded are not constant. This is because the overall statistical distributions are not quite Gaussian in shape. Of course, column 3 in Table 5 relied upon the availability of adequate statistical data. In the absence of such data column 2 alone would have had to suffice.

It is now clear that the percentage of time during which any single power-law sum will be exceeded, depends upon the form of the constituent or component statistical distributions and only convolution will supply an unambiguous answer. Of course, convolution reveals the need to specify the component distributions. It is felt that ambiguity will be reduced if, in the absence of exact knowledge of the component distributions, a rectangular shape is assumed and this is followed by convolution in order to select that combined tolerance which will be exceeded for a clearly defined percentage of time.

TABLE 5

Probabilities of Exceeding Three-Halves Power-Law Tolerances

1	2	3
Rectangular distributions to be convolved: $\pm x_M$	$\left(\sum x_M^{3/2}\right)^{2/3}$	Probability of exceeding ± values in Col. 2
5 * 10 * 3 * 4	14·6 ($\simeq 2{\cdot}06\sigma$)	2·8%
5 * 5 * 3 * 4	10·8 ($\simeq 2{\cdot}16\sigma$)	2·0%
5 * 5 * 3 * 6	12·1 ($\simeq 2{\cdot}15\sigma$)	2·0%
10 * 8 * 5 * 2	16·6 ($\simeq 2{\cdot}06\sigma$)	2·4%
10 * 5 * 5 * 2	14·8 ($\simeq 2{\cdot}05\sigma$)	2·4%
10 * 5 * 5 * 4	15·5 ($\simeq 2{\cdot}08\sigma$)	2·4%
10 * 10 * 8 * 5 * 2	21·5 ($\simeq 2{\cdot}17\sigma$)	3·6%

Of course, in considering in what units the errors should be measured, or rather plotted, when the rectangular histograms (of the random variations of error of each piece of apparatus in a television chain) are being established, complete freedom of choice exists. Some error units may be in decibels, some in degrees of phase angle, some in percentage of differential gain and so on. Obviously, only one kind of unit can be used in the calculation of one cumulative probability distribution from its constituent rectangular distributions. In certain instances it may be expedient to use power-law methods of addition before using the process of convolution. Thus, for example, it may be that the differential phase measured in one part of the chain occurs at a different luminance level from that at which it may be measured in another part of the chain. In this case, cancellation between negative and positive errors when the entire chain is subjected to an overall input-to-output measurement would not occur in a direct arithmetical sense. This difficulty could be approximately overcome by plotting the abscissae of the constituent rectangular distributions in, say, $\phi^{3/2}$, where ϕ is differential phase in degrees.*

To summarize: power-law methods are not a substitute for convolution, but they may be used to cover up inadequacies of measurement technique.

7. APPLICATIONS OF CONVOLUTION TO COLOUR TELEVISION

We may now apply the convolution technique to the performance figures set out in Table 3, but the results cannot be treated as more than an example of the way in which these matters will have to be dealt with in the future when more information becomes available. It is thought that a full statistical tolerance specification, even for a national colour television transmission, let alone for a transcontinental exchange, will be a quite complicated matter that may not be fully established for some years to come.

It is suggested that the broadcaster should take responsibility for the signal radiated to the viewers and that although an allowance must be made for receiver tolerances no account should be taken of the vagaries of propagation.

Take as the first example, differential phase. Leaving aside the receiver performance figure, Table 3, and assuming that a national distribution network including a contribution network, if any, can be maintained within a tolerance of ± 10° (in spite of the note in Table 3), we have to convolve rectangles as follows: 5°* 10°* 3°* 4°. Fig. 5 shows that, if we take 10% of the time as the time during which the tolerance will be exceeded, we arrive at the radiated signal tolerance of ± 11°. Note that for 5% of the time + 11° will be exceeded and for another 5% of the time the differential phase will be less than −11°. If the four performance figures used are now convolved with the receiver performance figure of 5° we have Fig. 6. From Fig. 6 we see that the final amount of differential phase that the viewer will suffer will for 10% of the time be numerically not less than 13°, but never numerically greater than 20°. The receiver manufacturing industry could, by studying these figures, decide whether simple-PAL receivers were a commercial proposition or not.

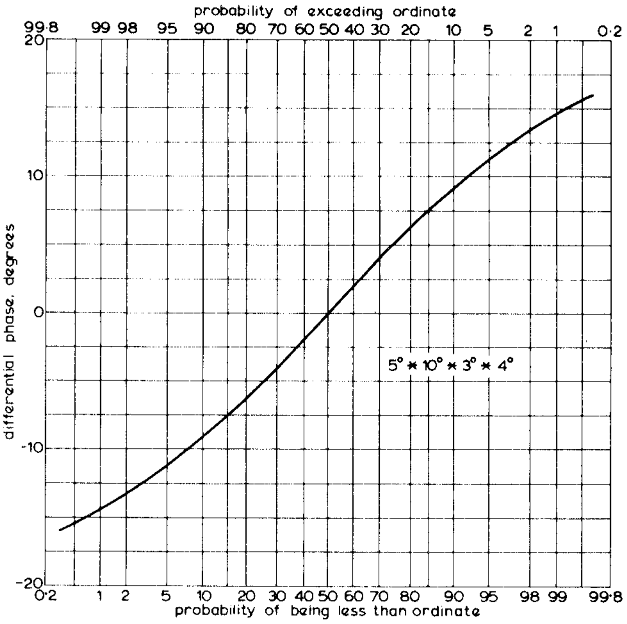

Fig. 5 - Differential phase: cumulative probability of errors in radiated signal

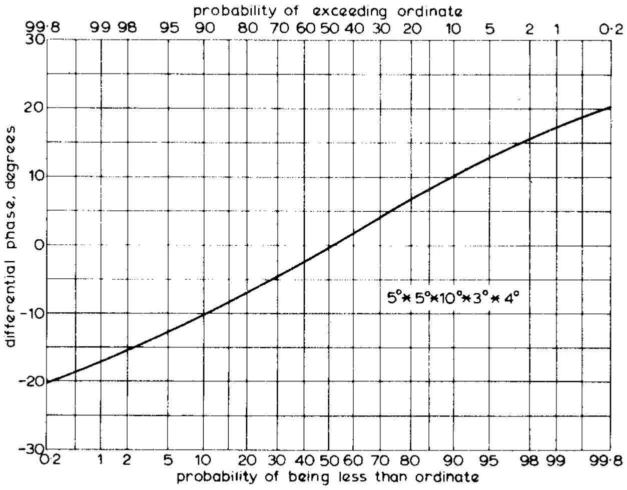

Fig. 6 - Differential phase: cumulative probability of overall errors

The second example will concern differential gain. Again taking the performance figures from Table 3, we have to convolve the following four rectangles: 10% * 8% * 5% * 2%. Fig. 7 shows that 13% will be exceeded numerically for 10% of the time. If we now convolve the above-mentioned four rectangles with the receiver performance figure of ± 10% we get Fig. 4(*a*) from which we see that for 10% of the time the viewer will suffer differential gain, numerically greater than 17%, but never numerically greater than 35%. Table 2 shows the overall tolerance for differential gain for grade 2·5A to be ± 30%; so at ± 17% or worse for 10% of the time we have a reasonably satisfactory state of affairs. Table 6 summarizes these results.

* Each error term of the resulting cumulative distribution would then have to be raised to the 2/3rds power.

TABLE 6

Tolerances not to be Exceeded for more than 10% of the Time

	Radiated signal	Overall figure	Overall tolerance (from Table 2) for grade 2·5A
Differential phase	± 11°	± 13°	± 40° (PAL_d)
Differential gain	± 13%	± 17%	± 30%

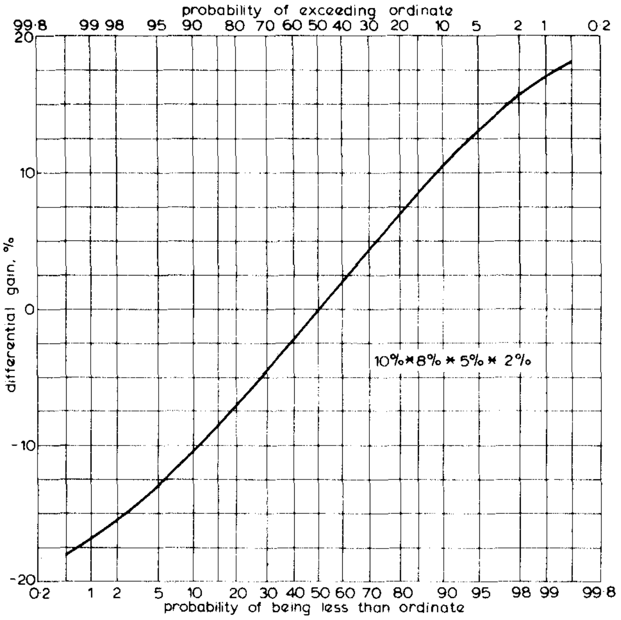

Fig. 7 - Differential gain: cumulative probability of errors in radiated signal

The same technique is recommended for the application of tolerances to the radiated signal, with respect to the ratio of the amplitude of the chrominance signal to that of the luminance signal and with respect to the timing of the chrominance signal relative to that of the luminance signal. In due course the same technique should be applied to all the important monochrome parameters such as, for example, the timings and durations of the synchronizing waveform.

8. CONCLUSIONS

What is proposed in this report is

(i) the use of the technique of convolution of errors rather than arbitrary rules of error addition which, although rapid, are approximate and do not permit a choice of a time-percentage-of-protection which ought to be chosen from considerations quite divorced from the technique of error addition employed. For example, the percentage of time during which errors exceeding certain values may be permitted should be determined from psychological or commercial considerations;

(ii) the determination of transmitted-signal tolerances taking into account not only the further distortions that may arise in the receiver, but also allotting to each segment of the transmission chain, component tolerances that are compromises between what can be achieved in practice and the contribution made by them to the overall figure of distortion.

9. REFERENCE

1. EBU. Report of the EBU Ad-hoc Group on Colour Television. Second edition, February 1965.

APPENDIX

In considering power-law methods of addition of errors in order to establish overall and component tolerances for a television chain, it is of interest to discuss briefly the question of statistical moments, because power-law addition is a special case of these.

The rth half-moment of a Gaussian distribution having a standard deviation, σ, is,

$$M_r = \int_0^\infty x^r e^{-x^2/2\sigma^2}\,dx \quad . \; . \; . \; . \; . \quad (1)$$

and if we normalize by dividing by the area M_0 where

$$M_0 = \sigma(\pi/2)^{1/2} \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

and we then take the rth root, we have

$$(M_r/M_0)_G^{1/r} = \sigma\sqrt{2}\left[\Gamma\{½(r+1)\}/\sqrt{\pi}\right]^{1/r} \quad (3)*$$

where the subscript "G" stands for "Gauss."

For $r = 3/2$ equation (3) gives

$$(M_r/M_0)_G^{1/r} = 0{\cdot}905\sigma \quad . \; . \; . \; . \quad (4)$$

For $r = 2$ equation (3) gives

$$(M_r/M_0)_G^{1/r} = \sigma$$

of course.

The area-normalized rth half-moment of a rectangular distribution is, of course,

$$\left(\frac{M_r}{M_0}\right)_R = \frac{x_M^r}{r+1} \quad . \; . \; . \; . \; . \; . \quad (5)$$

where "R" stands for "rectangle" and M_0 is, of course, x_M, the half-width of the rectangle.

$$\left(\frac{M_r}{M_0}\right)_R^{\frac{1}{r}} = \frac{x_M}{(r+1)^{1/r}} \quad . \; . \; . \; . \quad (6)$$

Note that the standard deviation of a rectangle is $x_M/\sqrt{3}$.

Equations (3) and (6) show that since the quantity

$$(M_r/M_0)^{1/r}$$

is in both cases proportional to σ we can, if we have a number of statistical distributions whose aleatory variables will add because they arise from tandem-connected apparatus, add the area-normalized half-moments to arrive at the area-normalized half-moment of the overall distribution. This is a generalization of the common practice of adding variances to obtain an overall variance. In electronic engineering, the latter process amounts to adding noise powers; from the above generalization, however, we could add any moments of the noise voltages if we should ever want to.

We are now in a position to consider the three-halves power law or the three-halves half-moment. We consider two cases:

(i) Gaussian distributions

Equation (4) shows that

$$(M_{3/2}/M_0)_G^{2/3} = 0{\cdot}9\sigma$$

Thus the probability of exceeding the two (positive and negative) three-halves half-moments is, from Fig. 4(*b*), about 33%. Since Gaussian distributions convolve into a Gaussian distribution this probability applies to the overall distribution and to its constituent distributions.

(ii) Rectangular distributions

This case is not quite so simple, because rectangular distributions do not convolve into a rectangular distribution. It is necessary to consider some examples in order to form an impression of the way things go. A number of rectangular distributions were convolved and one of the seven resulting overall cumulative distributions is shown in Fig. 4(*a*). Table 7, columns 1, 2 and 3 show the results of adding the area-normalized half-moments of the constituent rectangles and reading from the cumulative overall convolutions of the rectangles, the probabilities that the half-moments would be exceeded. The order of the moments was, of course, three-halves. Unlike the Gaussian case, we find that the probability, 46%, of exceeding the two three-halves moments of a single rectangle is not the same as that obtained from the convolution of several rectangles, although column 3 shows that there is not a great difference between the probabilities obtained from overall convolutions of four or five rectangles.

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Now, the actual method of using the three-halves power-law addition, in practice, differs from the foregoing method in that no assumption is made regarding the shape of statistical distributions. The simple rule adopted is to regard each component tolerance as an isolated number and these numbers are raised to the three-halves power, added, and the two-thirds power of the sum is then taken. This process is shown in columns 4 and 5 of Table 7. The difference between the numbers in column 5 is a greater proportion of them than was the case for the numbers in column 3.

TABLE 7

Probabilities of Exceeding Three-Halves Moments

1	2	3	4	5
Rectangular distributions to be convolved: $\pm x_M$	$\left(\sum \frac{x_M^{3/2}}{5/2}\right)^{2/3}$	Probability of exceeding ± values in Col. 2 (Note)	$\left(\sum x_M^{3/2}\right)^{2/3}$	Probability of exceeding ± values in Col. 4 (Note)
5 * 10 * 3 * 4	7·8 (=1·1σ)	26%	14·6 (=2·06σ)	2·8%
5 * 5 * 3 * 4	5·8 (=1·16σ)	24%	10·8 (=2·16σ)	2·0%
5 * 5 * 3 * 6	6·6 (1·17σ)	24%	12·1 (=2·15σ)	2·0%
10 * 8 * 5 * 2	9 (=1·12σ)	26%	16·6 (=2·06σ)	2·4%
10 * 5 * 5 * 2	8 (=1·11σ)	22%	14·8 (=2·05σ)	2·4%
10 * 5 * 5 * 4	8·4 (=1·19σ)	26%	15·5 (2·08σ)	2·4%
10 * 10 * 8 * 5 * 2	11·7 (=1·18σ)	25% (From Fig. 4(a))	21·5 (=2·17σ)	3·6%

Note: Obtained from cumulative curves of convolved rectangles given in Column 1.

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